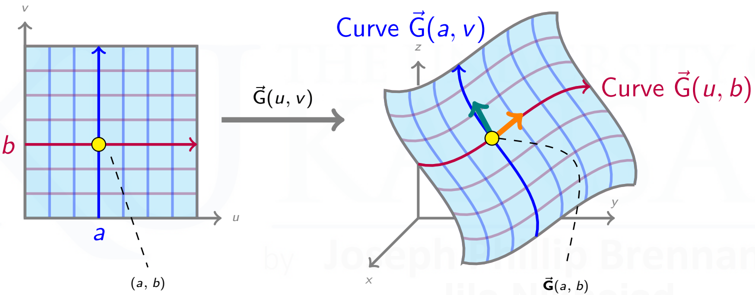
Section 16.5

Surface Integrals of Vector Fields

1 Tangent Lines and Planes of Parametrized Surfaces

Tangent Planes and Normal Vectors

Let $\mathcal{S}$ be a surface parametrized by $\vec{G}(u, v)$. Then $\vec{G}_u(a, b)$ and $\vec{G}_v(a, b)$ are tangent to the grid curves, thus span the tangent plane to $\mathcal{S}$ at P.



Normal vector: $\vec{N}(a, b) = \pm\vec{G}_u(a, b) \times \vec{G}_v(a, b)$

Unit normal vector: $\vec{n}(a, b) = \frac{\vec{N}(a,b)}{\|\vec{N}(a,b)\|}$

- The parametrization $\vec{G}$ is **regular** if $\vec{n}$ is well-defined ($\vec{N} \neq \vec{0}$ always).
- $\vec{n}$ and $-\vec{n}$ are two unit normal vectors; choose the correct orientation.

Tangent Planes and Normal Vectors

Tangent plane parametrization: $\vec{T}(r,s) = \vec{G}(a,b) + r\vec{G}_u(a,b) + s\vec{G}_v(a,b)$
Tangent plane equation: $\vec{N} \cdot ((x,y,z) - P) = 0$

Example 1: Find a parametrized equation and a Cartesian equation for the tangent plane to the helicoid $\vec{G}(u,v) = \langle u\cos(v), u\sin(v), v\rangle$ at $\vec{G}(1,0) = \langle 1,0,0\rangle$.

Solution:

$$\begin{aligned} \vec{G}_u &= \langle \cos(v),\ \sin(v),\ 0\rangle & \vec{G}_u(1,0) &= \langle 1,\ 0,\ 0\rangle \\ \vec{G}_v &= \langle -u\sin(v),\ u\cos(v),\ 1\rangle & \vec{G}_v(1,0) &= \langle 0,\ 1,\ 1\rangle \\ \vec{N} = \vec{G}_u \times \vec{G}_v &= \langle \sin(v),\ -\cos(v),\ u\rangle & \vec{N}(1,0) &= \langle 0,\ -1,\ 1\rangle \end{aligned}$$

Tangent plane parametrization:

$$\vec{T}(r,s) = (1,0,0) + r\langle 1,\ 0,\ 0\rangle + s\langle 0,\ 1,\ 1\rangle = \langle 1+r,\ s,\ s\rangle$$

Tangent plane intrinsic equation:

$$\langle 0,\ -1,\ 1\rangle \cdot ((x,y,z) - (1,0,0)) = 0 \qquad \text{or} \qquad -y + z = 0$$

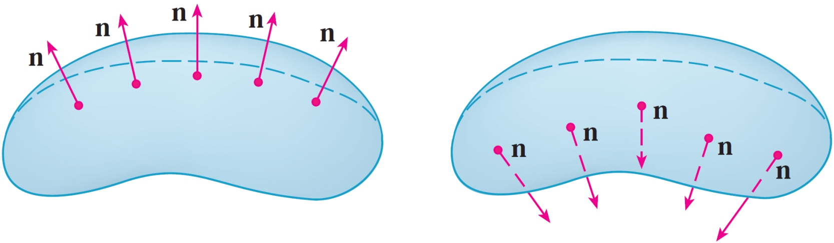
2 Oriented Surfaces

Flux and Orientation: Intuition

- Given a surface $\mathcal{S}$ and a vector field $\vec{F}$, we want to measure the **flux** (net flow of "stuff") of $\vec{F}$ through $\mathcal{S}$.
- In order for this to make sense, we need to specify **which way** stuff is flowing through $\mathcal{S}$.
- In other words, we need to choose one side of $\mathcal{S}$ as the "from" side and one side as the "to" side.
- In other words, we need an **orientation** for $\mathcal{S}$.
- In order to orient $\mathcal{S}$ *at a particular point*, choose one of the two unit normal vectors (which we think of as pointing from the "from" side to the "to" side).
- In order to measure flux through $\mathcal{S}$, we need this choice of $\vec{n}$ to be consistent for all points!

Precise Definition of Orientation

If it is possible to choose a unit normal vector $\vec{n}$ at every point so that $\vec{n}$ varies continuously over a surface $\mathcal{S}$, then $\mathcal{S}$ is called an **orientable surface** and the choice of $\vec{n}$ gives an **orientation** for $\mathcal{S}$.

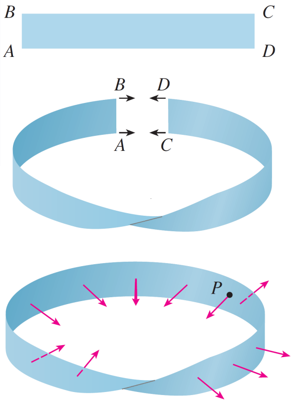


Any regular parametrization $\vec{G}(u, v)$ of an orientable surface automatically provides an orientation:

$$\vec{n} = \frac{\pm \vec{G}_u \times \vec{G}_v}{\|\vec{G}_u \times \vec{G}_v\|}$$

Not all surfaces can be oriented! The **Möbius strip** is a surface which has **only one side**, and is thus not orientable.

If an ant were to crawl along the Möbius strip starting at a point P, it would be upside down when it got back to P.



The Möbius strip can be parametrized as $\vec{G}(r,\theta) = (x(r,\theta), y(r,\theta), z(r,\theta))$, where

$$\begin{aligned} x &= 4\cos(\theta) + r\cos(\theta/2) && -1 \le r \le 1 \\ y &= 4\sin(\theta) + r\cos(\theta/2) && 0 \le \theta \le 2\pi \\ z &= r\sin(\theta/2) \end{aligned}$$

and it turns out (check for yourself!) that

$$\vec{G}(-1,0) = \vec{G}(1,2\pi)$$

but

$$\vec{N}(-1,0) = -\vec{N}(1,2\pi).$$

Closed Surfaces

A **closed surface** is the boundary of a solid region (e.g., spheres, ellipsoids, tori). Closed surfaces are always orientable (outward or inward).

Outward Orientation   Inward Orientation

Example 2: The parametrization $\vec{G}(\phi, \theta)$ of the unit sphere using spherical coordinates has outward orientation.

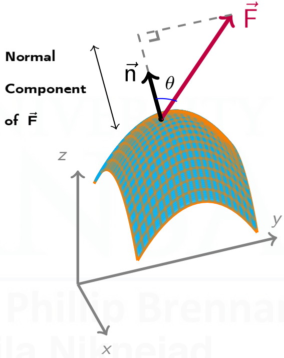
$$\begin{aligned}
\vec{G}(\phi,\, \theta) &= \langle \sin(\phi)\cos(\theta),\, \sin(\phi)\sin(\theta),\, \cos(\phi) \rangle \\
\vec{G}_\phi \times \vec{G}_\theta &= \left\langle \sin^2(\phi)\cos(\theta),\, \sin^2(\phi)\sin(\theta),\, \sin(\phi)\cos(\phi) \right\rangle \\
&= \sin(\phi)\, \vec{G}(\phi, \theta)
\end{aligned}$$

Vector Surface Integrals

Let $\mathcal{S}$ be an oriented surface with normal vector $\vec{n}$, and let $\vec{F}$ be a vector field.

The **normal component** of $\vec{F}$ with respect to $\mathcal{S}$ is $\vec{F} \cdot \vec{n}$.

This is a scalar-valued function on $\mathcal{S}$ that measures the extent to which $\vec{F}$ is flowing through $\mathcal{S}$ in the direction of $\vec{n}$.



The **vector surface integral** of $\vec{F}$ over $\mathcal{S}$ is

$$\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S} = \iint_{\mathcal{S}} \vec{F} \cdot \vec{n}\, dS.$$

3 Vector Surface Integrals and Flux

Vector Surface Integrals

If $\vec{F}$ is a continuous vector field defined on an oriented surface $\mathcal{S}$ with unit normal vector $\vec{n}$, then the **vector surface integral** of $\vec{F}$ over $\mathcal{S}$ is

$$\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S} = \iint_{\mathcal{S}} \vec{F} \cdot \vec{n}\, dS$$

The integral is also called the **flux** of $\vec{F}$ across $\mathcal{S}$.

If $\mathcal{S}$ has a regular parametrization $\vec{G}(u, v)$ over $\mathcal{R}$, then

$$\begin{aligned}
\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S} &= \iint_{\mathcal{S}} \vec{F} \cdot \vec{n}\, dS \\
&= \iint_{\mathcal{R}} \vec{F}\left(\vec{G}(u, v)\right) \cdot \frac{\pm \vec{G}_u \times \vec{G}_v}{\|\vec{G}_u \times \vec{G}_v\|} \|\vec{G}_u \times \vec{G}_v\|\, dA \\
&= \iint_{\mathcal{R}} \vec{F}\left(\vec{G}(u, v)\right) \cdot (\pm \vec{G}_u \times \vec{G}_v)\, dA \\
&= \iint_{\mathcal{R}} \vec{F}\left(\vec{G}(u, v)\right) \cdot \vec{N}\, dA
\end{aligned}$$

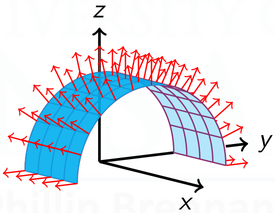
Vector Surface Integrals: Example

Example 3: Find the flux of $\vec{F}(x, y, z) = \langle 0, yz, z^2 \rangle$ outward through the surface $y^2 + z^2 = 4$, $z \geq 0$ between the planes $x = 0$ and $x = 1$.

Solution: The surface $\mathcal{S}$ is a half-cylinder, parametrized as

$$\vec{G}(x, \theta) = \langle x, 2\sin(\theta), 2\cos(\theta) \rangle$$

for $x \in [0, 1]$, $\theta \in [-\pi/2, \pi/2]$.



$$\vec{N} = \vec{G}_x \times \vec{G}_\theta = \langle 1, 0, 0 \rangle \times \langle 0,\ 2\cos(\theta),\ -2\sin(\theta) \rangle = \overbrace{\langle 0,\ 2\sin(\theta),\ 2\cos(\theta) \rangle}^{2\vec{n}}$$

Important Note: The orientation is outward (as intended), since $\vec{N}_z = 2\cos(\theta) \geq 0$ for $\theta \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

Example 3 (cont'd): $\vec{\mathsf{F}}(x,y,z) = \left\langle 0, yz, z^2 \right\rangle$ and $\vec{\mathsf{G}}(x,\theta) = \langle x, 2\sin(\theta), 2\cos(\theta)\rangle$ for $(x,\theta) \in [0,1] \times [-\pi/2, \pi/2]$.

$$\vec{\mathsf{F}}\left(\vec{\mathsf{G}}(x,\theta)\right) = \left\langle 0, 4\sin(\theta)\cos(\theta), 4\cos^2(\theta)\right\rangle$$

$$\vec{\mathsf{N}} = \langle 0,\ 2\sin(\theta),\ 2\cos(\theta)\rangle$$

By the formula for vector surface integrals,

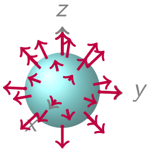
$$\iint_{\mathcal{S}} \vec{\mathsf{F}} \cdot d\vec{\mathsf{S}} = \iint_{\mathcal{R}} \vec{\mathsf{F}} \cdot \vec{\mathsf{N}}\ dA$$

$$= 8\int_{-\pi/2}^{\pi/2}\int_0^1 \underbrace{\sin^2(\theta)\cos(\theta) + \cos^3(\theta)}_{\cos(\theta)(\sin^2(\theta)+\cos^2(\theta))}\ dx\, d\theta$$

$$= 8\int_{-\pi/2}^{\pi/2} \cos(\theta)\ d\theta = 16.$$

▸ Video

Example 4: Find the flux of the vector field $\vec{F}(x,y,z) = \langle z, y, x \rangle$ across the unit sphere $x^2 + y^2 + z^2 = 1$, oriented outward.



Video

Solution: Parametrize the unit sphere as usual:

$$\vec{G}(\phi,\theta) = \langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi) \rangle = \vec{n} \qquad \phi \in [0,\pi], \;\; \theta \in [0, 2\pi]$$

$$\vec{G}_\phi \times \vec{G}_\theta = \overbrace{\langle \sin^2(\phi)\cos(\theta), \sin^2(\phi)\sin(\theta), \sin(\phi)\cos(\phi) \rangle}^{\sin(\phi)\vec{n}}$$

$$\vec{F}(\vec{G}(\phi,\theta)) = \langle \cos(\phi), \sin(\phi)\sin(\theta), \sin(\phi)\cos(\theta) \rangle$$

Then compute the vector surface integral:

$$\iint_{\mathcal{S}} \vec{F}\cdot d\vec{S} = \iint_{\mathcal{R}} \vec{F}\cdot\left(\vec{G}_\phi \times \vec{G}_\theta\right)\, dA$$

$$= \int_0^\pi \int_0^{2\pi} \left[2\sin^2(\phi)\cos(\phi)\cos(\theta) + \sin^3(\phi)\sin^2(\theta)\right]\, d\theta\, d\phi$$

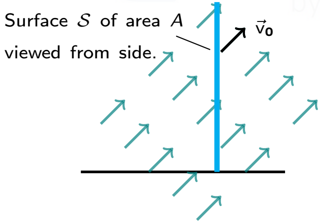
$$= \int_0^\pi 2\sin^2(\phi)\cos(\phi)\, d\phi \underbrace{\int_0^{2\pi} \cos(\theta)\, d\theta}_{0} + \underbrace{\int_0^{2\pi} \frac{1-\cos(2\theta)}{2}\, d\theta}_{\pi} \int_0^\pi \sin^3(\phi)\, d\phi = \frac{4\pi}{3}$$

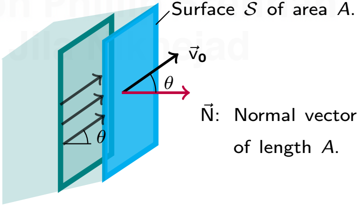
Fluid Flux

If $\vec{F}$ represents the **velocity** field of a fluid, then the **flow rate** across an oriented surface $\mathcal{S}$ is the vector surface integral

$$\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S}.$$

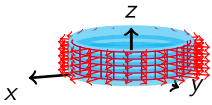
$$\text{Units} = (\text{units of } \vec{F}) \times (\text{units of } d\vec{S}) = \frac{\text{length}}{\text{time}} \times \text{area} = \frac{\text{volume}}{\text{time}}.$$





Example: Fluid Flux (Finding the Orientation!)

Example 5: A fluid flows with velocity $\vec{F}(x, y, z) = \langle z, y, x \rangle$ m/s, where x, y, z are measured in meters. Find the rate of flow outward through the cylinder $\mathcal{S}$ defined by $x^2 + y^2 = 4$ for $0 \le z \le 1$.



Solution: First, parametrize the cylinder: ▸ Video

$$\vec{G}(z, \theta) = \langle 2\cos(\theta),\ 2\sin(\theta),\ z \rangle \qquad \mathcal{R}:\ \theta \in [0, 2\pi],\ z \in [0, 1]$$

$$\vec{G}_z \times \vec{G}_\theta = \langle 0, 0, 1 \rangle \times \langle -2\sin(\theta),\ 2\cos(\theta),\ 0 \rangle$$

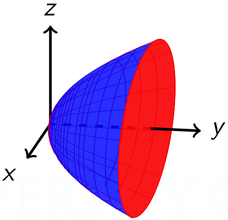
$$= \underbrace{\langle -2\cos(\theta),\ -2\sin(\theta),\ 0 \rangle}_{-2\vec{n}}$$

Note that G is oriented **inward** from $\mathcal{S}$. To fix this, just flip the sign of $\vec{N}$.

$$\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S} = \iint_{\mathcal{R}} \langle z,\ 2\sin(\theta),\ 2\cos(\theta) \rangle \cdot \underbrace{\langle 2\cos(\theta),\ 2\sin(\theta),\ 0 \rangle}_{\vec{N} = 2\vec{n}}\, dA$$

$$= \int_0^{2\pi} \int_0^1 2z\cos(\theta) + 4\sin^2(\theta)\, dz\, d\theta \ = \ 4\pi \text{ m}^3/\text{s}.$$

Example 6: Let $\mathcal{S}$ be the surface consisting of the paraboloid $y = x^2 + z^2$, $y \leq 1$ and the disk $x^2 + z^2 \leq 1$, $y = 1$, oriented outward. Find the flux of $\vec{F}(x, y, z) = \langle 0, y, -z \rangle$ through $\mathcal{S}$.



Solution: Call the paraboloid $\mathcal{P}$ and the disk $\mathcal{Q}$. Both can be parametrized over the domain $\mathcal{R}$ given by $0 \leq u \leq 1$, $0 \leq v \leq 2\pi$.

Parametrization of $\mathcal{P}$:

$$\vec{G}(u, v) = \left\langle u\cos(v),\ u^2,\ u\sin(v) \right\rangle$$

$$\vec{G}_u \times \vec{G}_v = \left\langle 2u^2\cos(v),\ -u,\ 2u^2\sin(v) \right\rangle$$

Parametrization of $\mathcal{Q}$:

$$\vec{H}(u, v) = \langle u\cos(v), 1, u\sin(v) \rangle$$

$$\vec{H}_u \times \vec{H}_v = \langle 0, -u, 0 \rangle$$

An outward normal to $\mathcal{S}$ should have negative y-coordinate on $\mathcal{P}$ and positive y-coordinate on $\mathcal{Q}$. Therefore, the normal vectors we want are

$$\vec{N}_{\mathcal{P}} = \vec{G}_u \times \vec{G}_v \qquad \vec{N}_{\mathcal{Q}} = -\vec{H}_u \times \vec{H}_v$$

Example 6 (continued):

$$\vec{G}(u,v) = \left\langle u\cos(v),\ u^2,\ u\sin(v)\right\rangle \qquad \vec{F}(\vec{G}(u,v)) = \left\langle 0,\ u^2,\ -u\sin(v)\right\rangle$$

$$\vec{H}(u,v) = \left\langle u\cos(v),\ u^2,\ u\sin(v)\right\rangle \qquad \vec{F}(\vec{H}(u,v)) = \langle 0,\ 1, -u\sin(v)\rangle$$

Recall that the normal vectors are

$$\vec{N}_{\mathcal{P}} = \left\langle 2u^2\cos(v),\ -u,\ 2u^2\sin(v)\right\rangle \qquad \vec{N}_{\mathcal{Q}} = \langle 0,\ u,\ 0\rangle$$

So the flux is

$$\iint_{\mathcal{S}} \vec{F}\cdot d\vec{S} = \iint_{\mathcal{P}} \vec{F}\cdot d\vec{S} + \iint_{\mathcal{Q}} \vec{F}\cdot d\vec{S}$$

$$= \iint_{\mathcal{R}} \vec{F}(\vec{G}(u,v))\cdot\vec{N}_{\mathcal{P}} + \vec{F}(\vec{H}(u,v))\cdot\vec{N}_{\mathcal{Q}}\, dA$$

$$= \int_0^{2\pi}\int_0^1 \left(-u^3 - 2u^3\sin^2(v) + u\right)\, du\, dv = 0$$

4 Summary, Comparing the Integrals

Summary: Types of Integrals

Let f be a scalar function and $\vec{F}$ a vector field.

Scalar Line Integral along a curve $\mathcal{C}$ parametrized by $\vec{r}(t)$ on $[a, b]$.

$$\int_{\mathcal{C}} f\, ds = \int_a^b f(\vec{r}(t))\, \|\vec{r}'(t)\|\, dt$$

Vector Line Integral along an oriented curve $\mathcal{C}$:

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

Scalar Surface Integral over a surface $\mathcal{S}$ parametrized by $\vec{G}(u, v)$ on $\mathcal{R}$:

$$\iint_{\mathcal{S}} f\, dS = \iint_{\mathcal{R}} f\left(\vec{G}(u, v)\right) \|\vec{G}_u \times \vec{G}_v\|\, dA$$

Vector Surface Integral over an oriented surface $\mathcal{S}$:

$$\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S} = \iint_{\mathcal{R}} \vec{F}\left(\vec{G}(u, v)\right) \cdot \left(\pm \vec{G}_u \times \vec{G}_v\right)\, dA$$

▸ Summary Video